

ON ALGEBRAIC AND QUANTUM RANDOM WALKS *

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Algebraic random walks (ARW) and quantum mechanical random walks (QRW) are investigated and related. Based on minimal data provided by the underlying bialgebras of functions defined on e. g the real line \mathbf{R} , the abelian finite group \mathbf{Z}_N , and the canonical Heisenberg-Weyl algebra \mathbf{hw} , and by introducing appropriate functionals on those algebras, examples of ARWs are constructed. These walks involve short and long range transition probabilities as in the case of \mathbf{R} walk, bistochastic matrices as for the case of \mathbf{Z}_N walk, or coherent state vectors as in the case of \mathbf{hw} walk. The increase of classical entropy due to majorization order of those ARWs is shown, and further their corresponding evolution equations are obtained. Especially for the case of \mathbf{hw} ARW, the diffusion limit of evolution equation leads to a quantum master equation for the density matrix of a boson system interacting with a bath of quantum oscillators prepared in squeezed vacuum state. A number of generalizations to other types of ARWs and some open problems are also stated. Next, QRWs are briefly presented together with some of their distinctive properties, such as their enhanced diffusion rates, and their behavior in respect to the relation of majorization to quantum entropy. Finally, the relation of ARWs to QRWs is investigated in terms of the theorem of unitary extension of completely positive trace preserving (CPTP) evolution maps by means of auxiliary vector spaces. It is applied to extend the CPTP step evolution map of a ARW for a quantum walker system into a unitary step evolution map for an associated QRW of a walker+quantum coin system. Examples and extensions are provided.

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1. Introduction

Random walks formulated in an algebraic framework^{1,2,3,4} of finite groups, bialgebras and operator algebras as well as in the framework of Quantum Mechanics^{5–13}, and references therein), are investigated. Minimal data for such constructions consist of a bialgebra¹⁴ and an integral (functional) defined on it, or alternatively of some Lie algebra, and two quantum systems modelling the walker and the coin system, together with a map modelling the coin tossing, that decides probabilistically the stepping of the walker.

Examples of ARWs treated in the following subsections are walks on algebras of functions on \mathbf{R} , on \mathbf{Z}_N , and on the canonical algebra Heisenberg-Weyl \mathbf{hw} ¹⁵ (section 2). For those walks we show how to define the entropy functional of their respective integral and/or Markov transition operator, and how to deduce that these are entropy increasing random walks by using arguments based on the interrelations relations between majorization bistochastic matrices and entropy^{16,17,18,19}. Moreover, a number theoretic decomposing of ARW on \mathbf{Z}_N is analysed, that is called prime decomposition²⁰, and refers to its factorization into products of similar smaller \mathbf{Z}_N -walks. Mathematically this decomposition is based on the Chinese Remainder Theorem and the co-associativity property. Also for the ARWs on \mathbf{R} , in addition to the usual case of short range walk with nearest neighbor (NN) transitions (Polya walk)²¹, we discuss in our algebraic framework the cases of: i) the NN centrally biased random walk (Gillis walk)²² and the case of symmetric random walk with exponentially distributed steps (Linderberg-Shuler:LS walk)²³. Finally, for the \mathbf{hw} ARW, where its functional is constructed by means of the eigenstates of the annihilation operator of that algebra i.e the family of coherent state vectors^{24,25}, the continuous time, or diffusion like limit, is obtained¹⁵. This limit results into a trace preserving quantum master equation^{26,27} for the density matrix of a quantum boson system, which physically is identified with the evolution equation of an open quantum boson system interacting coherently with a classical electric field and incoherently (dissipative interaction) with a bath of quantum oscillators *rigged* initially into a squeezed vacuum state (squeezed white noise)^{28,29,30}.

Section 3, gives a concise prescription of the concept of QRW, using the example of QRW on integers as paradigm¹³. It briefly explains the notion of quantum coin system and the coin tossing map, and summarizes two emblematic properties of that walk, namely the quadratic enhancement of its diffusion rate due to quantum entanglement between the walker and

coin systems, and the entropy increase without majorization effect of its probability distributions (pd). This section ends with a group theoretical scheme of classification of various known QRWs.

In section 4, a relation connecting ARW and QRW is put forward. The connection is grounded on the theorem due to Naimark that asserts the possibility of implementing in a unitary way a CPTP map operating on e.g the density operator of a quantum system³¹. This unitary extension is realized in the original vector space of the density operator augmented by an auxiliary vector space, the ancilla space in the terminology of Quantum Information theory³². Applied in the context of CPTP of a ARW, the ancilla space is identified with the quantum coin state space of a QRW. The Kraus generators determining the CPTP map of a ARW serve to built, albeit in non unique way, the unitary evolution operator of the associated QRW. This section concludes with the example of an explicit construction of a QRW associated to the **hw** ARW, of section 2.

Finally, section 4, summarized some of the results and gives some prospected applications of the ARW-QRW concepts and formalism.

2. Algebraic Random Walks

2.1. The case of R

Proposition 1. Let the bialgebra of real formal power series $H \equiv Fun(\mathbf{R})$ generated by the coordinate function X , and let the positive definite functional $\phi : H \rightarrow \mathbf{R}$, defined as $\phi = \sum_{i \in Z} p_i \phi_{\alpha_i}$, with $0 \leq p_i \leq 1$, $\sum_{i \in Z} p_i = 1$, where $\phi_{\alpha_i}(f(x)) = f(\alpha_i)$, the functional that evaluates any function $f \in H$, at the point $\alpha_i = i\alpha$, defined for some step $\alpha \in \mathbf{R}_+$, for $i \in Z$. The n -step convoluted functional becomes

$$\phi^{*n} = \sum_{i \in Z} p_i^{(n)} \phi_{\alpha_i} = \tilde{\phi}^T D^{n-1} p, \quad (1)$$

where $p^{(n)} = D^{n-1} p^{(1)}$, with the stochastic column vectors $p^{(k)} = (p_i^{(k)})_{i \in Z}$, $k = 1, 2, 3, \dots$ and initially $p^{(1)} \equiv p = (p_i)_{i \in Z}$. Also $D = (D_{ij}) = (p_{i-j})$, $i, j \in Z$ a bistochastic infinite matrix (delta matrix), and $\tilde{\phi} = (\phi_{\alpha_i})_{i \in Z}$ a column vector. Majorization ordering among pd's is valid at each step i.e $p^{(n+1)} \prec p^{(n)}$, and consequently the ARW is entropy increasing, namely, $S(\phi^{*n+1}) \geq S(\phi^{*n})$, $S(T_\phi^{n+1}) \geq S(T_\phi^n)$, where $S(\phi^{*n}) \equiv S(T_\phi^n) \equiv S(p^{(n)})$, with $S(x)$ any Shur-convex function e.g the classical entropy i.e $S(x) = -\sum_i x_i \log x_i$.

Proof: Operating with convoluted functionals on some function $f \in$

$H_{(33,34,35,36,37,38,39,40)}$ yields

$$\begin{aligned}\phi^{*2} &= (\phi \otimes \phi) \circ \Delta = \sum_{i,j \in Z} p_i^{(1)} p_j^{(1)} \phi_{\alpha_i + \alpha_j} \\ &= \sum_{k,j \in Z} p_i^{(1)} p_{k-i}^{(1)} \phi_{\alpha_k} \equiv \sum_{k,j \in Z} p_i^{(1)} D_{ki} \phi_{\alpha_k} = \tilde{\phi}^T D p.\end{aligned}\quad (2)$$

By induction we obtain the aimed relation $\phi^{*n} = \tilde{\phi}^T p^{(n)} = \tilde{\phi}^T D^{n-1} p^{(1)}$. Two important properties of the delta matrix are: i) *bistochasticity*, i.e the column and row sums is one, which is expressed by means of the column vector of units $e = (\dots, 1, 1, 1, \dots)$, that is left and right eigenvector of D , i.e $De = e$, $e^T = e^T D$, and ii) the *shift property* i.e $D_{i,j} = p_{i-j} = p_{i+1-(j+1)} = D_{i+1,j+1}$. This property in the case of ARW in Z governed by a pd with finite support, or in the case of a $N \subset Z$ finite dimensional walk (see remark below), amounts to a bistochastic matrix D .

The functional of the walk at each time step $\phi^{*n} = \sum_{i \in Z} p_i^{(n)} \phi_{\alpha_i}$, is characterized by the pd $p^{(n)} = (p_i^{(n)})_{i \in Z}$, which in turn is determined by the bistochastic matrix D i.e $p^{(n)} = D p^{(n-1)} = D^{n-1} p^{(1)}$. Let us assume that the pd's are of finite support (but see remark below), then by invocation of the theorem stating that two discrete pd's $x = (x_m)_m$, $y = (y_k)_k$, that are connected by a bistochastic matrix D i.e $x = Dy$, are ordered by majorization^{16,17,18,19} i.e $x \prec y$, we conclude that the sequence of pd's $\{p^{(1)}, p^{(2)}, p^{(3)}, \dots\}$, of site occupation probabilities resulting at each time step during the evolution of the walk is partially ordered by majorization i.e $p^{(n)} \prec p^{(n-1)}$, $n = 1, 2, 3, \dots$.

Let us adopt now the definition of the entropy of a functional to be the entropy of the pd that determines that functional i.e we set $S(\phi^{*n}) \equiv S(T_\phi^n) \equiv S(p^{(n)})$, or more generally we do so for any convex function $S : \mathbf{R} \rightarrow \mathbf{R}$ of the type of the so called Shur-convex functions e.g the classical Shannon entropy $S(p) = \sum_i p_i \log p_i$, or the functions $F(p) = \sum_i p_i^k$, for any constant $k \leq 1$, or $F(p) = -\prod_i p_i$ ¹⁷.

By virtue of the theorem stating that $x \prec y$ implies $F(x) < F(y)$, where $F(x) = \sum_i f(x_i)$, for any convex function $f : \mathbf{R} \rightarrow \mathbf{R}$, or otherwise said that the convex functions *isotonic* to majorization¹⁷, we conclude that for the pd resulting from the random walk the majorization ordering is valid at each step i.e. $p^{(n)} \prec p^{(n-1)}$, $n = 1, 2, 3, \dots$, and this implies ordering for e.g their entropies i.e. $S(p^n) > S(p^{n-1})$, and similarly for their functionals and transition operators. As majorization order implies entropy increase, it is considered as a measure of disorder, and this allow us to con-

clude the ARW are getting more disordered in the course of time with respect to their site-visiting pd's, which are getting more entropic, approaching, if left uninterrupted, to the uniform distribution of maximal entropy. \square

Remarks: 1) The assumption in the previous proof about the support of the pd's been finite is not actually necessary. In fact in the proof given by Hardy et. al ⁴¹ is stated that given $p \prec q$ for finite sequence of q, p pd's, use of Muirhead's algorithm leads to a bistochastic matrix A , such that $q = Ap$, and then the proposition: $H(Ap) \geq H(p)$ for H information/entropy or more generally a Shur convex function, is applied. This proof is constructive and builds A in a number of steps not greater than the lengths of p, q , therefore for infinite pd's a theorem not using bistochastic operators for the characterization of majorization is needed. Such a theorem is provided in ⁴².

2) For the above proposition the corresponding Markov transition operator defined as $T_\phi = (\phi \otimes id) \circ \Delta$, is equal to $T_\phi = \sum_{i \in \mathbf{Z}} p_i e^{\alpha_i \frac{d}{dx}}$. For the simplest case of $p_\pm = (p, 1 - p)$, $\alpha_\pm = \pm\alpha$, and all other p 's and α 's been zero, the continues limit $\lim_{n \rightarrow \infty} (T_\phi)^n \equiv T_\phi^\infty$ has been obtained that leads to a diffusion equation^{33,3,4,15}.

3) The above general algebraic setting implies that the stepping probability matrix $D = (D_{ij}) = (p_{i-j})$, $i, j \in \mathbf{Z}$, would be expanded in the enveloping algebra $\mathcal{U}(\mathfrak{e}(2))$ of the Euclidean Lie algebra $\mathfrak{e}(2) \approx \mathfrak{iso}(2)$, spanned by monomials of its generators $\{E_+, E_-, L\}$, that satisfy the defining commutation relations ⁴³

$$[E_+, E_-] = 0, \quad [L, E_\pm] = \pm E_\pm. \quad (3)$$

An irreducible matrix representation of those generators in the Hilbert space $\mathcal{H} \equiv l_2(\mathbf{Z})$ spanned by the eigenvectors of the "distance operator" L , is useful in expressing the D matrix for various random walks, and look for solutions by means of e.g the Fourier method. This irrep in the canonical basis of \mathcal{H} , and using the same symbol for abstract generators and their matrices reads:

$$L = \sum_{m \in \mathbf{Z}} m e_m e_m^\dagger, \quad E_\pm = \sum_{m \in \mathbf{Z}} e_{m \pm 1} e_m^\dagger. \quad (4)$$

Next we present three different random walks in \mathbf{R} , that can be used as show cases of the scheme of ARW presented here. In these concrete examples the defining the walk transition probability matrix D , is written as an element of the $\mathcal{U}(\mathfrak{e}(2))$ algebra. No attempt will be made to give an

algebraic solution for the problem of finding the n th-step site occupancy probability distribution, as this can be solved by other means.

The examples include: i) the simplest case of symmetric nearest-neighbor (NN) random walk (Polya-walk²¹); one of its deformations, ii) the NN centrally (site $n = 0$) biased random walk (Gillis-walk²²), which refers to a solvable case of a walk with no translational invariance, and stepping probabilities with a bias which has power law decay, or more specifically which decays in proportion from the origin of coordinates. The ε -deformation parameter is chosen so that when $\varepsilon > 0$, the walk is biased to enhance returns to the origin, while if $\varepsilon < 0$, escape from the origin is enhanced, and iii) a symmetric random walk with non- nearest-neighbor transitions with transition step length decaying according to an exponential law (LS-walk²³). Explicitly we have:

1) *Polya-walk*: symmetric nearest-neighbor(NN) random walk with

Markov transition operator

$$D_P = \frac{1}{2}(E_- + E_+) \quad (5)$$

with matrix elements the inter-site transition probabilities

$$D_P(l, l') = \frac{1}{2}\delta_{l, l'-1} + \frac{1}{2}\delta_{l, l'+1}. \quad (6)$$

2) *Gillis-walk*: nearest-neighbor centrally (site $n = 0$) biased random walk with Markov transition operator

$$D_G = D_G(\varepsilon) = \frac{1}{2}(E_- + E_+) + (E_- - E_+)\frac{\varepsilon}{2N}P_0^\perp, \quad -1 < \varepsilon < 1 \quad (7)$$

where $P_0 = e_0 e_0^\dagger$, $P_0^\perp = 1 - P_0$, the projection operators in the vector e_0 , and its orthogonal subspace respectively in $l_2(\mathbf{Z})$, and with matrix elements the inter-site transition probabilities

$$D_G(l, l') = \frac{1}{2}\left(1 + \frac{\varepsilon}{l'}\right)\delta_{l, l'-1} + \frac{1}{2}\left(1 - \frac{\varepsilon}{l'}\right)\delta_{l, l'+1}, \quad l' \neq 0 \quad (8)$$

$$D_G(l, 0) = \frac{1}{2}\delta_{l, -1} + \frac{1}{2}\delta_{l, 1}. \quad (9)$$

3) *LS-walk*: symmetric random walk with exponentially distributed steps with Markov transition operator

$$D_{LS} = D_{LS}(\varepsilon) = \frac{e^\varepsilon - 1}{2} \sum_{k \in \mathbf{Z}_{>0}} e^{-k\varepsilon} (E_-^k + E_+^k) \quad (10)$$

with matrix elements the inter-site transition probabilities

$$\begin{aligned} D_{LS}(l, l') &= \frac{e^\varepsilon - 1}{2} e^{-|l-l'|\varepsilon}, & |l-l'| > 0 \\ &= 0, & l = l'. \end{aligned} \quad (11)$$

Use of the eigenvector equations $E_\pm e = e$, $e^\dagger = e^\dagger E_\pm$, leads to the conclusion that the transition matrices of the Polya, and LS random walks i.e the matrices D_P , D_R , and D_{LS} respectively are bistochastic, while that of the Gillis walk D_G , is column stochastic. Also from the definitions the following two limits are deduced: $\lim_{\varepsilon \rightarrow 0} D_G(\varepsilon) = \lim_{\varepsilon \rightarrow \infty} D_{LS}(\varepsilon) = D_P$.

Generalizations: The 2D generalized Gillis random walk with entanglement. This model describes a 2D NN random walk with a biased towards a point placed at (m_1, m_2) coordinates on the plane. The bias is an attractive or repelling one depending on the sign of two parameters $(\varepsilon_1, \varepsilon_2)$ in reference to the motion along x, y axes respectively, and its strength decays following an inverse power law with characteristic exponents (a_1, a_2) correspondingly. This is summarized by writing explicitly all parameters in the Markov transition operator $D_G^{2D}(\varepsilon_1, m_1, a_1; \varepsilon_2, m_2, a_2)$, and their domain of values $\{\varepsilon_j, m_j, a_j\} \in \{(-1, 1), \mathbf{Z}, \mathbf{Z}_+\}$, $j = 1, 2$. The transition matrix along each axis is

$$D_G(\varepsilon_j, m_j, a_j) = \frac{1}{2}(E_- + E_+) + (E_- - E_+) \frac{\varepsilon_j}{2(N - m_j \mathbf{1})^{a_j}} P_j^\perp, \quad j = 1, 2 \quad (12)$$

where $P_j = e_j e_j^\dagger$, $P_j^\perp = 1 - P_j$, the projection operators in the vector e_j , and its orthogonal subspace respectively in $l_2(\mathbf{Z})$. The 2D transition operator consists of an entangled³² convex combination of two factorizable 1D transition operators with the parameters of position of bias site, characteristic decay exponent, and decay strength c.f. $\{m_j, a_j, \varepsilon_j\}$, $j = 1, 2$ interchanged, it reads ($0 \leq q \leq 1$)

$$\begin{aligned} D_G^{2D} &= q D_G(\varepsilon_1, m_1, a_1) \otimes D_G(\varepsilon_2, m_2, a_2) \\ &+ (1 - q) D_G(\varepsilon_2, m_2, a_2) \otimes D_G(\varepsilon_1, m_1, a_1), \end{aligned} \quad (13)$$

with matrix elements the inter-site transition probabilities

$$\begin{aligned}
D_G^{2D}(l_1, l'_1; l_2, l'_2) &= q \frac{1}{4} \left[1 + \frac{\varepsilon_1}{(l'_1 - m_1)^{a_1}} \right] \left[1 + \frac{\varepsilon_2}{(l'_2 - m_2)^{a_2}} \right] \delta_{l_1, l'_1-1} \delta_{l_2, l'_2-1} \\
&\quad + (1-q) \frac{1}{4} \left[1 + \frac{\varepsilon_2}{(l'_1 - m_2)^{a_2}} \right] \left[1 + \frac{\varepsilon_1}{(l'_2 - m_1)^{a_1}} \right] \delta_{l_1, l'_1-1} \delta_{l_2, l'_2-1},
\end{aligned} \tag{14}$$

with $(l'_1, l'_2) \neq (m_1, m_2)$, for the q component and $(l'_1, l'_2) \neq (m_2, m_1)$, for the $(1-q)$ component of the convex combination. Also

$$\begin{aligned}
D_G^{2D}(l_1, l'_1 = m_{1,2}; l_2, l'_2 = m_{2,1}) &= q \left(\frac{1}{2} \delta_{l_1, m_1-1} + \frac{1}{2} \delta_{l_1, m_1+1} \right) \\
&\quad \cdot \left(\frac{1}{2} \delta_{l_2, m_2-1} + \frac{1}{2} \delta_{l_2, m_2+1} \right) + (1-q) \left(\frac{1}{2} \delta_{l_1, m_2-1} + \frac{1}{2} \delta_{l_1, m_2+1} \right) \\
&\quad \cdot \left(\frac{1}{2} \delta_{l_2, m_1-1} + \frac{1}{2} \delta_{l_2, m_1+1} \right).
\end{aligned} \tag{15}$$

the matrix elements with $(l'_1, l'_2) = (m_1, m_2)$, for the q component and $(l'_1, l'_2) = (m_2, m_1)$, for the $(1-q)$ component of the convex combination. The role of bias and that of the entanglement of the two 1D walks, can be investigated in an algebraic manner in terms of tensor product representations of the $\mathbf{iso}(\mathbf{2})$ algebra, and it will be given elsewhere⁴⁴.

2.2. The case of \mathbf{Z}_N

In this section we give a brief study of algebraic random walks on abelian groups \mathbf{Z}_N ³, using their underlying bialgebra structure, and further investigate possible forms of their decomposition into simpler and dimensionally lower ARWs, based on number theoretic properties of N .

Proposition 2. Let the multiplicative abelian group $\mathbf{Z}_N = \{e, g, g^2, \dots, g^{N-1}\}$ and the bialgebra $H \equiv \mathbf{C}(\mathbf{Z}_N) \equiv Fun(\mathbf{Z}_N) = \mathbf{C} - span\langle \{g^i\}_{i=0}^{N-1} \rangle$, with dual algebra $H^* \equiv \mathbf{C}\mathbf{Z}_N \equiv Fun(\mathbf{Z}_N^*)$, with pairing given by evaluation. Let the positive definite functional (state) of a \mathbf{Z}_N random walk $\phi = \sum_{i \in [0, N-1]} p_i g^i$, identified as weighted sum of elements of H , with $0 \leq p_i \leq 1$, $\sum_{i \in [0, N-1]} p_i = 1$. By means of the column vectors $P = (p_i)_{i \in [0, N-1]}$, $G = (g^i)_{i \in [0, N-1]}$, we write it as $\phi = G^T P$, where T denotes transpose. Then the n -th step convolution becomes $\phi^{*n} = G^T D^{n-1} P$, where $D = (D_{ij}) = (p_{i-j \bmod N})$, $i, j \in [0, N-1]$ a circulant bistochastic matrix, to be called delta matrix.

Proof: Straightforward (c.f. ³).

Remarks: 0) The delta matrix is more precisely a circulant bistochastic matrix⁴⁵, that can be treated as an element of finite Heisenberg group H_N ²⁰, and this leads to an explicit solution for the dynamics of \mathbf{Z}_N walk.

1) Recall the following version of the Chinese Remainder Theorem (CRT)⁴⁶: let $N = N_1 N_2$, the decomposition of positive integer N , into product of coprimes N_1, N_2 , then working with the abelian additive groups of numbers $Z_{N_{1,2}}$, mod $N_{1,2}$ respectively, we can introduce the unique bi-jection

$$\delta : Z_N \longrightarrow Z_{N_1} \times Z_{N_2}, \quad \delta(x) = (x - \rho N_1, x - \sigma N_2), \quad (16)$$

$\rho, \sigma \in Z$, that maps the numbers of Z_N , into the ordered pair of its remainders after division by N_1, N_2 . Its dual map is the inverse $\mu \equiv \delta^{-1} = Z_{N_1} \times Z_{N_2} \longrightarrow Z_N$, which constructs the unique number x from its remainders (a, b) , with respect to the divisors N_1, N_2 , as $\mu(a, b) = a N_1^{\varphi(N_1)} + b N_2^{\varphi(N_2)} = x$, where $\varphi(c) = \# |c \geq m \in Z_+; GCD(m, c) = 1|$, the Euler function of given integer c , that equals the number of co-primes less or equal to c .

2) The same factorization is valid for abelian multiplicative groups i.e $\mathbf{Z}_N \cong \mathbf{Z}_{N_1} \otimes \mathbf{Z}_{N_2}$, if $N = N_1 N_2$, and N_1, N_2 , are relative primes.

3) Let for later use introduce now the map $\delta \longrightarrow V_\delta$, that uniquely determines from the CRT bijection δ , with $\delta(i) = (i_1, i_2)$, the isometric matrix $V_\delta : \mathbf{C}_N \longrightarrow \mathbf{C}_{N_1} \otimes \mathbf{C}_{N_2}$, written in the canonical basis as²⁰

$$V_\delta = \sum_{i \in Z_N} e_{\delta(i)} e_i^\dagger \equiv \sum_{i \in Z_N} (e_{i_1} \otimes e_{i_2}) e_i^\dagger, \quad (17)$$

and its inverse

$$V_\delta^\dagger = V_{\delta^{-1}} = \sum_{i_1 \in Z_{N_1}, i_2 \in Z_{N_2}} e_{\delta^{-1}(i_1, i_2)} e_{(i_1, i_2)}^\dagger \equiv \sum_{i_1 \in Z_{N_1}, i_2 \in Z_{N_2}} e_{\delta^{-1}(i_1, i_2)} (e_{i_1}^\dagger \otimes e_{i_2}^\dagger), \quad (18)$$

for which we have $V_\delta V_\delta^\dagger = V_\delta V_{\delta^{-1}} = V_{\delta \circ \delta^{-1}} = V_{id \times id} = \mathbf{1}_{\mathbf{C}_{N_1}} \otimes \mathbf{1}_{\mathbf{C}_{N_2}}$, and $V_\delta^\dagger V_\delta = V_{\delta^{-1}} V_\delta = V_{\delta^{-1} \circ \delta} = V_{id} = \mathbf{1}_{\mathbf{C}_N}$.

Notation: in the sequel and in order to distinguish the space dimensionality N , referring to certain e.g functional, operator, probability vector, co-multiplication etc, we will denote it by $\phi_{[N]}, T_{[N]}$, and $P_{[N]}$ respectively. We can now state two necessary and sufficient conditions, in order to obtain an isomorphic *prime decomposition* of a \mathbf{Z}_N ARW governed by a pd $P_{[N]}$, into a product of two others ARWs $\mathbf{Z}_{N_1}, \mathbf{Z}_{N_2}$, with respective pd's $P_{[N]} = P_{[N_1]} \otimes P_{[N_2]}$. The first condition is number theoretic and is about

the compositeness of the dimension number N of the probability distribution $P_{[N]}$ that generates the ARW on \mathbf{Z}_N , while the second condition is about its factorization into a tensor product of two pd's of appropriate dimensions.

Proposition 3. *i)* Let $N = N_1 N_2$ be the prime factorization of a positive integer N , then if in addition to the isomorphism of abelian groups of $Z_N \approx Z_{N_1} \otimes Z_{N_2}$, we consider a probability distribution (pd) $P_{[N]}$ factorizable into a product of two others pd's $P_{[N_1]}, P_{[N_2]}$, namely such that $V_\delta P_{[N]} = P_{[N_1]} \otimes P_{[N_2]}$, or $p_i^{[N]} = p_{i_1}^{[N_1]} p_{i_2}^{[N_2]}$, then the functional of a \mathbf{Z}_N algebraic random walk $\phi_{[N]} = \sum_{i \in Z_N} p_i^{[N]} g_{[N]}^i$, with $0 \leq p_i^{[N]} \leq 1$, $\sum_{i \in Z_N} p_i^{[N]} = 1$, factorizes for every step n , namely $\phi_{[N]}^{*n} \approx \phi_{[N_1]}^{*n} \otimes \phi_{[N_2]}^{*n}$, and similar factorization is valid for the transition operator i.e $T_{[N]}^n \approx T_{[N_1]}^n \otimes T_{[N_2]}^n$.

ii) Let N_1, N_2 and N_3 , be the co-prime factors of some positive integer N , then the decomposition $Z_N \approx Z_{N_1} \otimes Z_{N_2} \otimes Z_{N_3} \approx Z_{N_1 N_2} \otimes Z_{N_3} \approx Z_{N_1} \otimes Z_{N_2 N_3}$, is co-assosiative¹⁴ as indicated in the last two equations, and if a pd $P_{[N]}$ is considered for which the finest decomposition $V_\delta P_{[N]} = P_{[N_1]} \otimes P_{[N_2]} \otimes P_{[N_3]}$, in terms of three others pd's is true, then the associated \mathbf{Z}_N ARW is also decomposed at any step n , namely for its functional and transition operator respectively, the following co-assosiative factorizations are valid

$$\phi_{[N]} \approx \phi_{[N_1]} \otimes \phi_{[N_2]} \otimes \phi_{[N_3]} \approx \phi_{[N_1 N_2]} \otimes \phi_{[N_3]} \approx \phi_{[N_1]} \otimes \phi_{[N_2 N_3]}, \quad (19)$$

Proof: *i)* The $N = N_1 N_2$ case: By means of the factorization property

of the pd viz.

$$\begin{aligned} V_\delta P_{[N]} &= \sum_{i \in Z_N} p_i^{[N]} V_\delta e_i \approx \sum_{i \in Z_N} p_{\delta(i)}^{[N]} e_{\delta(i)} = \sum_{i_1 \in Z_{N_1}, i_2 \in Z_{N_2}} p_{(i_1, i_2)}^{[N]} e_{i_1} \otimes e_{i_2} \\ &= \sum_{i_1 \in Z_{N_1}, i_2 \in Z_{N_2}} p_{i_1}^{[N_1]} p_{i_2}^{[N_2]} e_{i_1} \otimes e_{i_2} = P_{[N_1]} \otimes P_{[N_2]}, \end{aligned} \quad (20)$$

we deduce that

$$\begin{aligned} V_\delta D_{[N]} V_\delta^\dagger &= \sum_{i, j \in Z_N} p_{(i-j)}^{[N]} V_\delta e_i e_j^\dagger V_\delta^\dagger \\ &\approx \sum_{i_1 \in Z_{N_1}, i_2 \in Z_{N_2}} p_{\delta(i-j)}^{[N]} e_{i_1} \otimes e_{i_2} \cdot e_{i_1}^\dagger \otimes e_{i_2}^\dagger = D_{[N_1]} \otimes D_{[N_2]}, \end{aligned} \quad (21)$$

and therefore $V_\delta D_{[N]}^n V_\delta^\dagger \approx D_{[N_1]}^n \otimes D_{[N_2]}^n$, for the delta matrix. This yields

the factorization of the functional

$$\begin{aligned}\phi_{[N]} &= G_{[N]}^T P_{[N]} = G_{[N]}^T V_{\delta}^{\dagger} V_{\delta} P_{[N]} \\ &\approx (V_{\delta} G_{[N]}^T)^{\dagger} (V_{\delta} P_{[N]}) = G_{[N_1]}^T \otimes G_{[N_2]}^T \cdot P_{[N_1]} \otimes P_{[N_2]} = \phi_{[N_1]} \otimes \phi_{[N_2]}\end{aligned}$$

and similarly for the transition operator (with $\tau(x \otimes y) = y \otimes x$)

$$\begin{aligned}T_{[N]} &= (id \otimes \tau \otimes id) \circ (\phi_{[N]} \otimes id_{[N]}) \circ \Delta_{[N]} \\ &\approx (\phi_{[N_1]} \otimes id_{[N_1]} \otimes \phi_{[N_2]} \otimes id_{[N_2]}) \circ (\Delta_{[N_1]} \otimes \Delta_{[N_2]}) \approx T_{[N_1]} \otimes T_{[N_2]}. \quad (23)\end{aligned}$$

ii) The $N = N_1 N_2 N_3$ case: Applying the CRT to N , in two different ways yields for the abelian group Z_N two factorizations i.e $Z_N \approx Z_{N_1} \otimes Z_{N_2} \otimes Z_{N_3} \approx Z_{N_1 N_2} \otimes Z_{N_3} \approx Z_{N_1} \otimes Z_{N_2 N_3}$. This can equivalently be expressed by means of the bijection δ , as

$$(\delta_{[N_1, N_2]} \otimes id) \circ \delta_{[N_1 N_2, N_3]} = (id \otimes \delta_{[N_2, N_3]}) \circ \delta_{[N_1, N_2 N_3]}. \quad (24)$$

Performing the factorization of the pd $P_{[N]}$, as before twice we have

$$(V_{\delta_{[N_1, N_2]}} \otimes \mathbf{1}) V_{\delta_{[N_1 N_2, N_3]}} P_{[N]} = P_{[N_1]} \otimes P_{[N_2]} \otimes P_{[N_3]} \quad (25)$$

which implies that

$$(V_{\delta_{[N_1, N_2]}} \otimes \mathbf{1}) V_{\delta_{[N_1 N_2, N_3]}} D_{[N]}^n ((V_{\delta_{[N_1, N_2]}} \otimes \mathbf{1}) V_{\delta_{[N_1 N_2, N_3]}})^{\dagger} \approx D_{[N_1]}^n \otimes D_{[N_2]}^n \otimes D_{[N_3]}^n. \quad (26)$$

This factorization of the delta matrix leads to an equivalent factorizations of the functional at each time step of the \mathbf{Z}_N walk, i.e

$$\phi_{[N]}^{*n} \approx \phi_{[N_1 N_2]}^{*n} \otimes \phi_{[N_3]}^{*n} \approx \phi_{[N_1]}^{*n} \otimes \phi_{[N_2 N_3]}^{*n} \approx \phi_{[N_1]}^{*n} \otimes \phi_{[N_2]}^{*n} \otimes \phi_{[N_3]}^{*n} \quad (27)$$

Similar decompositions can be obtained for Markov transition operators.

□

Remark: Interpretation of this result implies that the original random walk \mathbf{Z}_N , is isomorphically decomposed into the product of : i) three similar walks $\mathbf{Z}_N \cong \mathbf{Z}_{N_1} \otimes \mathbf{Z}_{N_2} \otimes \mathbf{Z}_{N_3}$, ii) or into the product of two walks of dimensions $N_1 N_2$ and N_3 i.e $\mathbf{Z}_{N_1 N_2} \otimes \mathbf{Z}_{N_3}$ and iii) or into the product of two walks of dimensions N_1 and $N_2 N_3$ i.e $\mathbf{Z}_{N_1} \otimes \mathbf{Z}_{N_2 N_3}$. As an example we take the \mathbf{Z}_6 random walk on the vertices of a canonical hexagon the statistics of which is determined by a 6-dimension pd vector $P_{[6]}$. If this pd has been chosen so that there are two other pd's one 3-dimensional $P_{[3]}$ that generates a \mathbf{Z}_3 ARW on the vertices of a canonical triangle, and one 2-dimensional $P_{[2]}$ that generates a \mathbf{Z}_2 walk on two points, such that $P_{[6]} = P_{[3]} \otimes P_{[2]}$, then we can decompose the ARW

on the canonical hexagon as a product of two walks one on the triangle times one on the two-point set. Similar interpretations can be given to the prime decomposition of an ARW on a canonical polygon. E.g. the decomposition $\mathbf{Z}_{60} \cong \mathbf{Z}_3 \otimes \mathbf{Z}_4 \otimes \mathbf{Z}_5 \cong \mathbf{Z}_{12} \otimes \mathbf{Z}_5 \cong \mathbf{Z}_3 \otimes \mathbf{Z}_{20}$ ²⁰.

Problem: Let $N = N_1 N_2$ be the prime factorization of a positive integer N , for which the isomorphism $Z_N \approx Z_{N_1} \otimes Z_{N_2}$, is valid, let us consider an \mathbf{Z}_N ARW generated as previously by a pd $P_{[N]}$. If this pd is not factorizable into a product of two others pd's as in the remark above, but instead there are two pairs of pd's $P_{[N_1]}$, $P_{[N_1]}^{\prime}$, and $P_{[N_2]}$, $P_{[N_2]}^{\prime}$, such that the original pd is a convex combination of them i. e. ($0 \leq q \leq 1$),

$$V_\delta P_{[N]} = q P_{[N_1]} \otimes P_{[N_2]} + (1 - q) P_{[N_1]}^{\prime} \otimes P_{[N_2]}^{\prime}. \quad (28)$$

In the terminology of quantum information³², we have here two ARWs $\mathbf{Z}_{N_1}, \mathbf{Z}_{N_2}$, that are *classically correlated (cc)* and form a probabilistic decomposition of the ARW in \mathbf{Z}_N , and write symbolically $\mathbf{Z}_N \cong q \mathbf{Z}_{N_1} \otimes \mathbf{Z}_{N_2} + (1 - q) \mathbf{Z}_{N_1}^{\prime} \otimes \mathbf{Z}_{N_2}^{\prime}$. A number of interesting problems arise in this context: the total dynamics and reduced dynamics of the components of the \mathbf{Z}_N walk; the problem of construction of measures of correlations among the components of \mathbf{Z}_N walk; the problem of information(majorization) dynamics and information exchange among the pd's components of the \mathbf{Z}_N walk; the problem of asymptotics of the \mathbf{Z}_N walk.

2.3. The case of hw-Algebra

A bialgebra¹⁴ $\mathcal{A} = \mathcal{A}(\mu, \eta, \Delta, \epsilon)$ over a field k is a vector space equipped with an algebra structure with homomorphic associative product map $\mu : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, and a homomorphic unit map $\eta : k \rightarrow \mathcal{A}$, that are related by $\mu \circ (\eta \otimes id) = id = \mu \circ (id \otimes \eta)$, together with a coalgebra structure with a homomorphic coassociative coproduct map $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ and a homomorphic counit map $\epsilon : \mathcal{A} \rightarrow k$, that are related between them by $(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta$. Both products satisfy the compatibility condition of bialgebra i.e. $(\mu \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) = \Delta \circ \mu$, where $\tau(x \otimes y) = y \otimes x$ stands for the twist map. If η or ϵ is not defined in \mathcal{A} we speak about non unital or non counital Hopf algebra.

Suppose we have a functional $\phi : \mathcal{A} \rightarrow \mathbf{C}$, defined on \mathcal{A} , let us define the operator $T_\phi : \mathcal{A} \rightarrow \mathcal{A}$ as $T_\phi = (\phi \otimes id) \circ \Delta$, then $\epsilon \circ T_\phi = \phi$, namely the counit aids to pass from the operator to its associated functional. From this relation we can define the convolution product $\psi * \phi$, between functionals

as follows ³³:

$$\begin{aligned}
 \epsilon \circ T_\psi T_\phi &= \epsilon \circ (\psi \otimes id) \circ \Delta \circ (\phi \otimes id) \circ \Delta \\
 &= (\phi \otimes \psi) \circ (id \otimes id \otimes \epsilon) \circ (id \otimes \Delta) \circ \Delta \\
 &= (\phi \otimes \psi) \circ \Delta = \phi * \psi,
 \end{aligned} \tag{29}$$

and in general $\epsilon \circ T_\phi^n = \epsilon \circ T_{\phi^{*n}} = \phi^{*n}$. These last relations imply that the transition operators form a discrete semigroup with respect to their composition with identity element $T_\epsilon \equiv id$ (due to the axioms of bialgebra) and generator T_ϕ , while the functionals form a dual semigroup with respect to the convolution with identity element e and generator ϕ , and that these two semigroups are homomorphic to each other.

We recall now the *Heisenberg-Weyl algebra* **hw** and its structural maps: this is the algebra of the quantum mechanical oscillator and is generated by the creation, annihilation and the unit operator $\{a^\dagger, a, \mathbf{1}\}$ respectively which satisfy the commutation relation (Lie bracket) $[a, a^\dagger] = \mathbf{1}$, while $\mathbf{1}$ commutes with the other elements. This algebra possesses a non counital bialgebra structure ³, chapt. 3), with comultiplication defined as

$$\begin{aligned}
 \Delta^{(n-1)} a &= n^{-\frac{1}{2}} (a \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes a \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \cdots \otimes a), \\
 \Delta^{(n-1)} a^\dagger &= n^{-\frac{1}{2}} (a^\dagger \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \cdots \otimes a^\dagger), \\
 \Delta \mathbf{1} &= \mathbf{1} \otimes \mathbf{1},
 \end{aligned} \tag{30}$$

where as indicated above the $\Delta^{(n-1)}$ maps the creation/annihilation operators into the n th fold tensor product of the algebra and adds appropriate factors (also c.f. ⁴⁷). Let us also define the number operator $N = a^\dagger a$ with the following commutation relations with the generators of **hw**: $[N, a^\dagger] = a^\dagger$, $[N, a] = -a$. The module which carries the unique irreducible and infinite dimensional representation of the oscillator algebra is the Hilbert-Fock space \mathcal{H}_F which is generated by a lowest (or "vacuum") state vector $|0\rangle \in \mathcal{H}_F$ and is given as $\mathcal{H}_F = \{|n\rangle = \frac{(a^\dagger)^n}{n!} |0\rangle, n \in \mathbf{Z}_+\}$.

The functionals we intend to use will be defined by means of the canonical coherent state vectors of the **hw** algebra so in the sequel we give a brief introduction to the concept of coherent state vectors (CSV) on Lie groups: consider a Lie group \mathcal{G} , with a unitary irreducible representation $T(g)$, $g \in \mathcal{G}$, in a Hilbert space \mathcal{H} . We select a reference vector $|\Psi_0\rangle \in \mathcal{H}$, to be called the "vacuum" state vector, and let $\mathcal{G}_0 \subset \mathcal{G}$ be its isotropy subgroup, i.e for $h \in \mathcal{G}_0$, $T(h)|\Psi_0\rangle = e^{i\varphi(h)}|\Psi_0\rangle$. The map from the factor group $\mathcal{M} = \mathcal{G}/\mathcal{G}_0$ to the Hilbert space \mathcal{H} , introduced in the form of an orbit of the vacuum state under a factor group element, defines a CSV

$|x\rangle = T(\mathcal{G}/\mathcal{G}_0)|\Psi_0\rangle$ labelled by points $x \in \mathcal{M}$ of the coherent state manifold. Coherent states form an (over)complete set of states, since by means of the Haar invariant measure of the group \mathcal{G} viz. $d\mu(x)$, $x \in \mathcal{M}$, they provide a resolution of unity, $\mathbf{1} = \int_{\mathcal{M}} d\mu(x) |x\rangle\langle x|$. As a consequence, any vector $|\Psi\rangle \in \mathcal{H}$ is analyzed in the CS basis, $|\Psi\rangle = \int_{\mathcal{M}} d\mu(x) \Psi(x) |x\rangle$, with coefficients $\Psi(x) = \langle x|\Psi\rangle$. We should note here that the square integrability of the vectors Ψ will impose some limits on the growth parameters of the functions $\Psi(x)$ at the boundary of manifold \mathcal{M} (cf. ^{24,25} and references therein).

The **hw**-CS is defined by the relation

$$|\alpha\rangle = e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle = \mathcal{N} e^{\alpha a^\dagger} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (31)$$

It is an (over)complete set of normalized states with respect to the measure $d\mu(\alpha) = \frac{1}{\pi} e^{-|\alpha|^2} d^2\alpha$ for the non-normalized CS, and $\alpha \in \mathcal{M} = HW/U(1) \approx \mathbf{C}$ is the CS manifold. Since $a|\alpha\rangle = \alpha|\alpha\rangle$, \mathcal{M} is the flat canonical phase plane with the standard line element $ds^2 = d\alpha d\bar{\alpha}$. Also the symplectic 2-form $\omega = i d\alpha \wedge d\bar{\alpha}$ is associated to the canonical Poisson bracket $\{f, g\} = i(\partial_\alpha^f \partial_{\bar{\alpha}}^g - \partial_{\bar{\alpha}}^f \partial_\alpha^g)$.

The density operator (state) ρ which would be used to determine functionals of some operator bialgebras \mathcal{A} , is defined generally as follows : Let a Hilbert vector space \mathcal{H} that carries a unitary irreducible representation of \mathcal{A} of finite or infinite dimension. The set of density operators

$$\mathcal{S} = \{\rho \in \text{End}(\mathcal{H}) : \rho \geq 0, \rho^\dagger = \rho, \text{tr} \rho = 1\}, \quad (32)$$

namely the set of non-negative, Hermitian, trace-one operators acting on \mathcal{H} form a convex subspace of $\text{End}(\mathcal{H})$, which is the convex hull of the set

$$\mathcal{S}_P = \{\rho \in \mathcal{S}, \rho^2 = \rho\} \equiv \mathcal{H}/U(1), \quad (33)$$

namely of the set of pure density operators (states), that are in one-to-one correspondence with the state vectors of \mathcal{H} .

The density operator to be used in the case of **hw** walk uses the pure density operators $|\pm\alpha\rangle\langle\pm\alpha| \in \mathcal{S}_P$ and is a convex combination belonging to the convex hull of \mathcal{S}_P i.e $0 \leq p \leq 1$,

$$\rho = p |\alpha\rangle\langle\alpha| + (1-p) |-\alpha\rangle\langle-\alpha|. \quad (34)$$

Let $\phi(\cdot) = \text{Tr} \rho(\cdot) \equiv \langle \rho, \cdot \rangle$, a functional defined on the enveloping Heisenberg-Weyl algebra $\mathcal{U}(hw)$, where $\rho = p |\alpha\rangle\langle\alpha| + (1-p) |-\alpha\rangle\langle-\alpha|$, i.e the ρ density operator is given as a convex sum of pure state density

operators. The action of the transition operator $T_\phi = (\phi \otimes id) \circ \Delta$ on the generating monomials of $\mathcal{U}(hw)$ (where we ignore the numerical factors in the comultiplication of eq.(30)) reads,

$$\begin{aligned} T_\phi((a^\dagger)^m a^n) &= (\phi \otimes id) \circ \Delta((a^\dagger)^m a^n) \\ &= \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} [p\alpha^{*i}\alpha^j + (1-p)(-\alpha)^i(-\alpha)^j](a^\dagger)^{m-i}a^{n-j} \\ &= p(a^\dagger + \alpha^*)^m(a + \alpha)^n + (1-p)(a^\dagger - \alpha^*)^m(a - \alpha)^n. \end{aligned} \quad (35)$$

For a general element $f(a, a^\dagger) \in \mathcal{U}(hw)$ that is normally ordered, namely the annihilation operator a is placed to the right of the creation operator a^\dagger , denoted by $f(a, a^\dagger) = \sum_{m,n \geq 0} c_{mn}(a^\dagger)^m a^n$, the action of the linear operator T_ϕ becomes

$$T_\phi(f(a, a^\dagger)) = pf(a + \alpha, a^\dagger + \alpha^*) + (1-p)f(a - \alpha, a^\dagger - \alpha^*) \quad (36)$$

By means of the CS eigenvector property and the normal ordering of the f element we also compute the value of functional viz.

$$\phi(f(a, a^\dagger)) = pf(\alpha, \alpha^*) + (1-p)f(-\alpha, -\alpha^*). \quad (37)$$

Let us consider the displacement operator $D_\alpha = e^{\alpha a^\dagger - \alpha^* a} \equiv e^\Lambda$, with $\Lambda = \alpha a^\dagger - \bar{\alpha} a$ which acts with the group adjoint action on any element f of the $\mathcal{U}(hw)$ algebra viz.²⁴

$$AdD_\alpha(f) = Ade^{\alpha a^\dagger - \alpha^* a}(f) = e^{ad(\alpha a^\dagger - \alpha^* a)}(f) = e^{ad\Lambda} = D_\alpha f D_\alpha^\dagger, \quad (38)$$

where $ad(X)f = [X, f]$ and $ad(X)ad(X)f = [X, [X, f]]$ and similarly for higher powers, stands for the Lie algebra adjoint action that is defined in terms of the Lie commutator. Similarly the group adjoint action in terms of the displacement operator on the generators of $\mathcal{U}(hw)$ reads $AdD_{\pm\alpha}(a) = a \mp \alpha$ and $AdD_{\pm\alpha}(a^\dagger) = a^\dagger \mp \alpha^*$. By means of these expressions we rewrite the action of the transition operator as

$$\begin{aligned} T_\phi(f(a, a^\dagger)) &= [pAdD_{-\alpha} + (1-p)AdD_\alpha]f(a, a^\dagger) \\ &= [pe^{-ad\Lambda} + (1-p)e^{ad\Lambda}]f(a, a^\dagger). \end{aligned} \quad (39)$$

Next we compute the limiting transition operator

$$\begin{aligned} T_t \equiv T_{\phi_t} &\equiv \lim_{n \rightarrow \infty} T_\phi^n = \\ &\lim_{n \rightarrow \infty} \left[1 + \frac{t}{n} ad\Lambda + \frac{t}{n} \gamma (ada^\dagger)^2 + \frac{t}{n} \bar{\gamma} (ada)^2 - \right. \\ &\quad \left. \frac{t}{n} |\gamma| (ada \, ada^\dagger + ada^\dagger \, ada) \right]^n. \end{aligned} \quad (40)$$

In the last expression we have introduced the parameters of continuous time $t \in \mathbf{R}$ and the drift and diffusion terms respectively $c, \gamma \in \mathbf{C}$ by means of the relations

$$2\alpha(p - \frac{1}{2}) = \frac{tc}{n}, \quad \frac{a^2}{2} = \frac{t\gamma}{n}, \quad (41)$$

and have performed the limits $\alpha \rightarrow 0, n \rightarrow \infty$, with t, c, γ been fixed. We have also been used the limit $\lim_{n \rightarrow \infty} (1 + \frac{Z}{n}) = e^Z$, to obtain the continuous time Markov transition operator $T_t = e^{tL}$, with its L generators as it is obviously identified in the equation below

$$T_t = e^{tL} \equiv \exp t [1 + ad\Lambda + \gamma(ada^\dagger)^2 + \bar{\gamma}(ada)^2 - |\gamma| (ada ada^\dagger + ada^\dagger ada)] . \quad (42)$$

By construction T_t is the time evolution operator for any element f of $\mathcal{U}(hw)$ i.e $f_t = T_t(f)$ and forms a continuous semigroup $T_t T_{t'} = T_{t+t'}$ under composition. This yields the diffusion equation obeyed by f_t , which will be taken to be normally ordered hereafter. By time derivation of the equation

$$\phi_t(f) = \langle \rho, f_t \rangle = \langle \rho, e^{tad\mathcal{L}} f \rangle = \langle e^{-tad\mathcal{L}^\dagger} \rho, f \rangle = \langle \rho_t, f \rangle , \quad (43)$$

we obtain the diffusion equation $\frac{d}{dt} f_t = \mathcal{L} f_t$, as well as the dual one satisfied by the ρ density operator viz. $\frac{d}{dt} \rho_t = \mathcal{L}^\dagger \rho_t$.

Explicitly the quantum master evolution equation for the density matrix reads

$$\begin{aligned} \frac{d}{dt} \rho(t) = & [ca^\dagger - \bar{c}a, \rho] + \gamma(a^{\dagger 2} \rho + \rho a^\dagger - 2a^\dagger \rho a^\dagger) + \bar{\gamma}(a^2 \rho + \rho a^2 - 2a \rho a) \\ & - |\gamma| ((2N+1) \rho + \rho(2N+1) - 2a^\dagger \rho a - 2a \rho a^\dagger). \end{aligned} \quad (44)$$

The obtained equation is similar to the quantum master equation that describes the trace preserving dynamics of the reduced density matrix operator of single mode of the electromagnetic field interacting coherently with classical electric field while it is immersed in a bath of quantum oscillators²⁸. The decay of the field mode is influenced by the kind of initial condition the reservoir oscillators are put in. To analyze the physical content of that equation we rewrite it below by separating its right hand side into three lines i.e

$$\begin{aligned} \frac{d}{dt} \rho(t) = & [ca^\dagger - \bar{c}a, \rho] \\ & - |\gamma| (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger + a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a) \\ & + \gamma(a^{\dagger 2} \rho + \rho a^\dagger - 2a^\dagger \rho a^\dagger) + \bar{\gamma}(a^2 \rho + \rho a^2 - 2a \rho a). \end{aligned} \quad (45)$$

The first line gives the coherent interaction of the mode with the classical electric field of intensity c , as described by the commutator of density operator with the Hamiltonian term. It is neglected for balanced walk. The second line is a typical part of a master equation describing mode decaying for reservoir oscillators in thermal equilibrium⁴⁸. The last line is related to the case where the reservoir is prepared in a squeezed vacuum state²⁸. Closing we should notice that the above quantum master equation can be transformed into a Fokker-Planck partial differential equation for some quasi-probability function e.g P, Q , or Wigner function associated with the density operator e.g²⁸.

3. Quantum Random Walks

3.1. The case of Z

In a quantum random walk there are two dynamically coupled systems: the walker systems described by a Hilbert space H_w , and the coin system also described by a 2D Hilbert space $H_c \approx \mathbf{C}^2 = \text{span}(|+\rangle, |-\rangle)$. Let the unitary matrix U operating in H_c , e.g $U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, or $U_{\frac{\pi}{4}} = \sigma_3 U_H$, the Hadamard (Fourier) transform and the $\frac{\pi}{4}$ -rotation matrix respectively. If we denote by $P_{\pm} = |\pm\rangle\langle\mp|$, the projection of an orthogonal partition of H_c , and by S_{\mp} , two step operators in the walker's space H_w (explicit examples determined below), we introduce the unitary one-step evolution operator acting in the space $H_c \otimes H_w$ in the combined coin+walker system:

$$V = \sum_{m=\pm} P_m U \otimes S_m = \frac{1}{\sqrt{2}} \begin{pmatrix} S_+ & S_+ \\ \pm S_- & \mp S_- \end{pmatrix}. \quad (46)$$

In the equation above the upper/lower signs correspond to the choices $U_H, U_{\frac{\pi}{4}}$, respectively. If initially the two systems are decoupled, their density matrices are factorized i.e $\varrho_c \otimes \varrho_w$. Then if we assume that $\rho_c = |\Psi\rangle\langle\Psi|$, is a projective density matrix with $|\Psi\rangle = a|+\rangle + b|-\rangle$, a normalized coin state vector, then the 1-time step of the QRW is considered as a completely positive trace (CPTP) map ε_v , operating on the walkers' density operator, obtained by partially tracing out ("forgetting") the coin system i.e

$$\varepsilon_V(\varrho_w) = \text{Tr}_c(V \varrho_c \otimes \varrho_w V^\dagger) = \sum_{m=\pm} p_m S_m \varrho_w S_m^\dagger. \quad (47)$$

Unitarity of V , implies that the two probabilities $p_{\pm} = \text{Tr} P_{\pm} U \rho_c U^{\dagger} P_{\pm} = \frac{1}{2} \pm 2 \text{Re} a \bar{b} \geq 0$, $p_+ + p_- = 1$, are determined by the coin system state vector variables and in turn they determine the Kraus generator³¹, $(\sqrt{p_+} S_+, \sqrt{p_-} S_-)$, of CPTP evolution map. Four sources of choices are implicit in the above prescription of QRW: the choice of the initial coin+walker state vectors, the choice of unitary U , the choice of definition of time step in terms of the tracing of the coin system (to be investigated in detail below), and finally the choice of step operators S_{\mp} in the walker's space, that determines the king of the QRW under investigation.

Next we choose to turn to a Hadamard random walk on integers with dynamical algebra the Euclidean algebra $\mathfrak{e}(2) \approx \mathfrak{iso}(2)$, with step operators $(\sqrt{p_+} S_+, \sqrt{p_-} S_-) = (\sqrt{p} E_+, \sqrt{(1-p)} E_-)$, and third element the "distance" operator L . The latter is the interesting quantum observable the quantum moments of which are used to compare classical and quantum walks. We examine three possible tracing schemes: i) the classical scheme that *promptly traces the coin system* after each V action, leading to the CRW

$$\varepsilon_V(\rho_w^{(n)}) = \text{Tr}_c(V \rho_c \otimes \rho_w^{(n-1)} V^{\dagger}) = p E_+ \varrho_w^{(n-1)} E_+^{\dagger} + (1-p) E_- \varrho_w^{(n-1)} E_-^{\dagger}, \quad (48)$$

which produces the diagonal sequence of density matrices

$$\left\{ \rho_w^{(0)}, \varepsilon_V(\rho_w^{(0)}), \varepsilon_V^2(\rho_w^{(0)}), \varepsilon_V^3(\rho_w^{(0)}), \varepsilon_V^4(\rho_w^{(0)}), \dots \right\}, \quad (49)$$

with diagonal elements the probabilities of site occupancy, given by the $(p, 1-p)$ classical Pascal triangle, ii) the scheme that *traces the coin system by increasing delays* i.e. after an increasing number of actions of V operator, leading to a walk designated by QRW₁

$$\varepsilon_{V^N}(\rho_w^{(0)}) = \text{Tr}_c(V^N \rho_c \otimes \rho_w^{(0)} V^{\dagger N}) = \sum_{m=\pm} S_m^{(N)} \varrho_w^{(0)} S_m^{(N)\dagger}, \quad (50)$$

which produces the sequence of density matrices

$$\left\{ \rho_w^{(0)}, \varepsilon_V(\rho_w^{(0)}), \varepsilon_{V^2}(\rho_w^{(0)}), \varepsilon_{V^3}(\rho_w^{(0)}), \varepsilon_{V^4}(\rho_w^{(0)}), \dots \right\}, \quad (51)$$

iii) and the scheme of *delaying the trace of coin system by exactly one action of V operator*, leading to a walk designated by QRW₂

$$\varepsilon_{V^2}(\rho_w^{(n)}) = \text{Tr}_c(V^2 \rho_c \otimes \rho_w^{(n-1)} V^{2\dagger}) = B_+ \varrho_w^{(n-1)} B_+^{\dagger} + B_- \varrho_w^{(n-1)} B_-^{\dagger}, \quad (52)$$

which produces the sequence of density matrices

$$\left\{ \rho_w^{(0)}, \varepsilon_{V^2}(\rho_w^{(0)}), \varepsilon_{V^2}^2(\rho_w^{(0)}), \varepsilon_{V^2}^3(\rho_w^{(0)}), \varepsilon_{V^2}^4(\rho_w^{(0)}), \dots \right\}. \quad (53)$$

Next proposition deals with the time evolution of the pd $(P_q^{(N)})_m =$

$\langle m | \rho_w^{(N)} | m \rangle$, made of the diagonal elements of the density matrices in the course of the walk QRW₁ and QRW₂¹³.

Remarks: 1) From the above treatment can be shown that the QRW₁ reproduces along the diagonal elements of the sequence of evolving density matrices of the walker system the pd of the 1D Hadamard random walk and its diffusion rate¹³ 2) by expressing the evolution unitary operator V , as the exponential of a hermitian operator H i.e $V = e^{iH}$, which describes an interaction between coin and walker quantum systems, the various schemes of partial tracing can be physically implemented by choosing the length of interaction time e.g the QRW₂ scheme requires an interaction time $t = 2$ for the evolution operator $V_t = \exp(itH)$.⁴⁴

Proposition 4. There exists bistochastic matrices $\Delta_q = S_0 \circ \overline{S}_0 + S_1 \circ \overline{S}_1$, and $\Delta_q = B_0 \circ \overline{B}_0 + B_1 \circ \overline{B}_1$, which determine the pd drawn from the diagonal elements of the evolving density matrices of the walks QRW₁, and QRW₂ respectively, by means of the respective equations

$$P_q^{(n+1)} = \Delta_c P_q^{(n)} + (E_+ - E_-) M^{(n)} P_q^{(0)}, \quad (54)$$

$$P_q^{(n+1)} = \Delta_q P_q^{(n)}, \quad (55)$$

where

$$M^{(n)} = \sqrt{p(1-p)} S_0^{(N)} \circ \overline{S}_1^{(N)} + S_1^{(n)} \circ \overline{S}_0^{(n)} + (2p-1) S_0^{(n)} \circ \overline{S}_0^{(n)}, \quad (56)$$

and

$$\begin{aligned} \Delta_q = & \left| pa + \sqrt{p(1-p)}b \right|^2 E_+^2 + \left| \sqrt{p(1-p)}a - pb \right|^2 E_-^2 \\ & + \left| (1-p)a - \sqrt{p(1-p)}b \right|^2 + \left| (1-p)b + \sqrt{p(1-p)}a \right| \mathbf{1}. \end{aligned} \quad (57)$$

Above the element by element or Hadamard product $\Delta = A \circ B$, defined between matrices A, B of the same size by $(A \circ B)_{ij} = A_{ij} B_{ij}$, has been used¹⁷. The study of the pd obtained from the previous proposition reveals two novel aspects of the models QRW₁ and QRW₂. First the aspect of breaking the condition *majorization-implies- entropy increase*, and that of *enhanced diffusion rates*. Before closing this section we give a brief demonstration of the latter one (detail investigation together with relevant references can be found in ¹³).

Enhanced Diffusion Rates: Let us consider the m th order statistical moment of the distance operator $\langle L^m \rangle_n = \text{Tr}(\rho_w^{(n)} L^m)$, at the n th step. Assume we have a symmetric walk with $\langle L \rangle_n = 0$, so that the standard deviation at n th step is $\sigma_n = \sqrt{\langle L^2 \rangle_n}$. For the CRW we have that $\sigma_n^C = \sqrt{n}$, and from the pd's taken by the previous proposition we obtain the respective standard deviation for the walks QRW₁ and QRW₂, expressed in terms of their classical counterparts, for the first five steps: $\{\sigma_1^{QRW_1} = \sigma_1^C, \sigma_2^{QRW_1} = \sigma_2^C, \sigma_3^{QRW_1} = \sigma_3^C, \sigma_4^{QRW_1} = \sqrt{5}/2 \sigma_4^C, \sigma_5^{QRW_1} = \sqrt{8/5} \sigma_5^C\}$, and $\{\sigma_1^{QRW_2} = \sigma_1^C, \sigma_2^{QRW_2} = \sqrt{5}/2 \sigma_2^C, \sigma_3^{QRW_2} = \sqrt{3} \sigma_3^C, \sigma_4^{QRW_2} = \sqrt{7/2} \sigma_4^C, \sigma_5^{QRW_2} = 2 \sigma_5^C\}$.

From that we deduce that there is a quadratic speed up of the spreading rate in the case of QRW₁ with respect to CRW, and that rate is even bigger for the case of QRW₂. Finally the asymptotic ($n \gg 1$), growth values are turn out to be $\sigma_n^{QRW_1} \sim \sqrt{n(2 - \sqrt{2})/2} \sigma_n^C$.

4. Relation of Algebraic and Quantum Random Walks

This section will put forward a relation between the two types of random walks under investigation so far i.e the ARQ and QRW. The relevant theory here is Naimark's extension theorem that allows to express in a non unique manner a positive trace preserving map, operating by means of its Kraus generators on a density matrix describing the state of some quantum system in a certain Hilbert space, by a unitary operator acting on a extension of the original space.

Stated in the language of random walks the extension theorem assumes a ARW described by a CPTP map $\varepsilon(\varrho_w) = \sum_{m=\pm} p_m S_m \varrho_w S_m^\dagger$, operating on the density matrix of *walker system* with its Kraus generators $(\sqrt{p_+} S_+, \sqrt{p_-} S_-)$, defined to act on Hilbert space H_w . It is further assumed as usually for ARWs, that the step generators S_\pm are related to an algebra of operators that needs to be specified. Then a unitary operator V is considered acting on $H_c \otimes H_w$ i.e an extension of the original space by an extra or ancilla space $H_c \approx \mathbf{C}^2 = \text{span}(|+\rangle, |-\rangle)$, which in the context of QRW stands for the *coin system*. Let a pure density matrix in the coin system $\rho_c = |\Psi\rangle\langle\Psi|$. Then the extension theorem provides a unitary representation of the CPTP i.e

$$\varepsilon_V(\varrho_w) = \sum_{m=\pm} p_m S_m \varrho_w S_m^\dagger = \text{Tr}_c(V \varrho_c \otimes \varrho_w V^\dagger). \quad (58)$$

The unitary operator provides the Kraus generators as $\sqrt{p_m} S_m = \langle m| V |\Psi\rangle$, up to a local unitary operator W , i.e the transformation $V \rightarrow$

$W \otimes \mathbf{1}_w V$, provides the same generators. For the case of ARQs the unitary operator V is specifically expressed by means of the coin states projections P_{\pm} , and the unitary matrix $U(p_{\pm}) = \begin{pmatrix} \sqrt{p_+} & \sqrt{p_-} \\ \sqrt{p_-} & -\sqrt{p_+} \end{pmatrix}$ of the coin space, as $V = \sum_{m=\pm} P_m U \otimes S_m$. In particular the step operators are unitary and inverse to each other i.e $S_+ = (S_-)^{\dagger}$. This unitary representation can be extended to products of CPTM maps by first defining the unitaries

$$V^{\otimes 2} \equiv \sum_{m,n=\pm} P_m U \otimes P_n U \otimes S_m S_n. \quad (59)$$

Then we obtain

$$\varepsilon_V^2(\varrho_w) = \sum_{m,n=\pm} p_m p_n S_m S_n \varrho_w (S_m S_n)^{\dagger} = Tr_c \otimes Tr_c (V \otimes V \varrho_c \otimes \varrho_c \otimes \varrho_w V^{\dagger} \otimes V^{\dagger}). \quad (60)$$

This unitary extension of ε^2 , requires a double ancilla space or two coin quantum systems coupled, so the total space is $H_c \otimes H_c \otimes H_w$. In the general case the k th power of the CPTP map of an ARW can be implemented unitarily by extending the original *walker space* by k ancillary coin systems, so the total space becomes $H_c^{\otimes k} \otimes H_w$, and the total unitary operator is

$$V^{\otimes k} \equiv \sum_{m_1, \dots, m_k=\pm} P_{m_1} U \otimes \dots \otimes P_{m_k} U \otimes S_{m_1} \dots S_{m_k} \quad (61)$$

with unitarity condition $V^{\otimes k} V^{\otimes k \dagger} = \mathbf{1}_c^{\otimes k} \otimes \mathbf{1}_w$. Then we obtain for the k th step of the QRW as described by k successive actions of its CPTP map, a unitary realization which involves tensoring of *quantum walker* to k quantum *coin systems*, followed by a coupling of them by a unitary evolution operator on the space of coins+walker composite system, and finally a decoupling of coins from the walker system, taken by partially tracing with respect to the coin Hilbert spaces. The partial tracing corresponds to coin tossing in an ordinary random walk, and results into a density matrix for the quantum system of the walker, which further may provide statistics of various quantum observables of the walk.

Explicitly the unitarization of ARW reads

$$\begin{aligned} \varepsilon_V^k(\varrho_w) &= \sum_{m_1, \dots, m_k=\pm} p_{m_1} \dots p_{m_k} S_{m_1} \dots S_{m_k} \varrho_w (S_{m_1} \dots S_{m_k})^{\dagger} \\ &= Tr_c^{\otimes k} (V^{\otimes k} \varrho_c^{\otimes k} \otimes \varrho_w V^{\otimes k \dagger}), \end{aligned} \quad (62)$$

and can be identified with a QRW.

Let us remark at this point that an equivalent decomposition of the unitary $V^{\otimes k}$ would be

$$V^{\otimes k} = \prod_{i=1}^k W_i, \text{ where } W_i = (P_+ U)_i \otimes (S_+)_{k+1} + (P_- U)_i \otimes (S_-)_{k+1}, \quad (63)$$

and the subindex denotes the position of the embedding of the respective operator into the k -fold tensor product. In fact each of these operators $W_i \in \text{End}(H_c^{\otimes k} \otimes H_w)$, provide a new decomposition of the CPTP map i.e $\varepsilon_V^k = \varepsilon_{W_1 W_2 \dots W_k}$, which is equivalent to a nonstationary QRW with k different unitary evolution operators employed in order to construct the k -th step. As a matter of fact this new decomposition helps to account for the type of quantum entanglement involved between coin and walker systems. Let us take the simplest $k = 2$ case, where $V^{\otimes 2} = W_1 W_2$, with

$$\begin{aligned} W_1 &= (P_+ \otimes \mathbf{1}_c \otimes S_+ + P_- \otimes \mathbf{1}_c \otimes S_-) U \otimes \mathbf{1}_c \otimes \mathbf{1}_w, \\ W_2 &= (\mathbf{1}_c \otimes P_+ \otimes S_+ + \mathbf{1}_c \otimes P_- \otimes S_-) \mathbf{1}_c \otimes U \otimes \mathbf{1}_w. \end{aligned} \quad (64)$$

The action of these operators on the product density matrices $\rho_c \otimes \rho_c \otimes \rho_w$ is akin to the action of some control-control- S_{\pm} type of non-local operator, which uses the two coin states as *control spaces* and the walker state as the *target space*, preceded by the local unitary operator U which acts on the control spaces and creates appropriate superposition of coin states. These actions generate quantum entanglement and can be described by the quantum circuit of Fig.1.c below. For purpose of comparison in Fig.1.a, we have included the corresponding circuit that generates the four entangled bipartite Bell states upon action of the composite operator $U_{CN} H \otimes \mathbf{1}$ on the four orthogonal product qubit states, and on Fig. 1b the circuit that corresponds to the unitary $V = (P_+ \otimes S_+ + P_- \otimes S_-) U \otimes \mathbf{1}$, of the ε_V map.

Though the topic of the entanglement in the QRWs will not be investigated further here, it should be obvious from the above analysis that the CPTP map that implements some discrete time-step of a QRW, also generates quantum entanglement that can be studied by appropriate circuits and evaluated by effective measures, as usually is done in other cases of coupled quantum systems. Although quantum correlations have been generally accepted to be the common cause of all novel effects in the QRW performance, the exact evaluation of the entanglement resources needed in the course of a QRW is still an open problem.

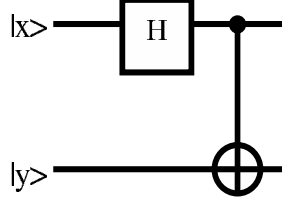


Fig.1.a

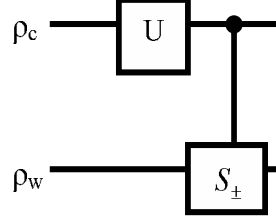


Fig.1.b

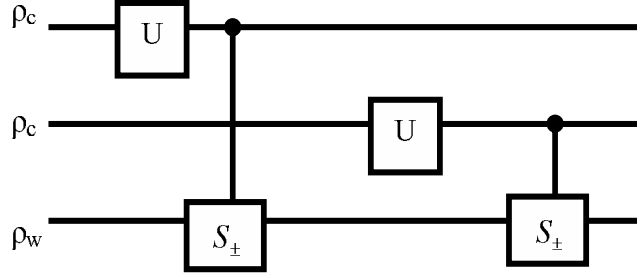


Fig.1.c

Figure 1. Fig.1 (a) Circuit generating entangled Bell states from product states using local Hadamard and non-local control-not gate; (b) Circuit generating the unitary evolution of the 1-step map ε_V ; (c) Circuit generating entangled coin-walker states from corresponding factorized ones using local unitary (e.g Hadamard gate in the case of the homonymous QRW), and non-local control-control- S_{\pm} gates, that form the map of the 2-step QRW $\varepsilon_V^2 = \varepsilon_{W_1 W_2}$.

To establish further the connection among ARQs and QRWs we give four particular examples:

i) the Euclidean QRW ¹³ [**iso(2)**-QRW]: with the *distance operator* $L|m\rangle = m|m\rangle$, with its eigenspace $H_w^L = \text{span}\{|m\rangle, m \in \mathbf{Z}\}$, and its dual *phase operator* $\Phi|\varphi\rangle = \varphi|\varphi\rangle$, with its eigenspace $H_w^\Phi = \text{span}\{|\varphi\rangle, \varphi \in [0, 2\pi); \frac{d\varphi}{2\pi}\}$, related by a Fourier transform with H_w^L . Two ARWs and its associated QRWs (modulo local unitary operators in coin spaces, as explained above), can be constructed: the *distance* random walk on \mathbf{Z} , with CPTP map constructed with Kraus generators been the step operators in the distance operator eigenstates i.e $S_{\pm}|m\rangle \equiv E_{\pm}|m\rangle = e^{\pm i\Phi}|m\rangle = |m \pm 1\rangle$, and

the *phase* random walk on the circle S , with CPTP map constructed with Kraus generator been the step operators in the phase operator eigenstates i.e $S_{\pm} |\varphi\rangle \equiv e^{\pm iL} |\varphi \pm 1\rangle$;

ii) the Canonical Algebra QRW [**hw**-QRW]: with the *position operator* $Q |q\rangle = q |q\rangle$, with its eigenspace $H_w^Q = \text{span}\{|q\rangle, q \in \mathbf{R}; dq\}$, and its dual *momentum operator* $P |p\rangle = p |p\rangle$, with its eigenspace $H_w^P = \text{span}\{|p\rangle, p \in \mathbf{R}; dp\}$, related by a Fourier transform with H_w^Q . Two ARWs and its associated QRWs (modulo local unitary operators in coin spaces, as explained above), can be constructed: the *position* random walk on \mathbf{R} , with CPTP map constructed with Kraus generators been the step operators in the position operator eigenstates i.e $S_{\pm} |q\rangle \equiv e^{\pm iP} |q\rangle = |q \pm 1\rangle$, and the *momentum* random walk on \mathbf{R} , with CPTP map constructed with Kraus generator been the step operators in the momentum operator eigenstates i.e $S_{\pm} |p\rangle \equiv e^{\pm iQ} |p\rangle = |p \pm 1\rangle$;

iii) the M - dimensional Discrete Heisenberg Group QRW [**h_M**-QRW]: with the *action operator* $N |n\rangle = n |n\rangle$, with its eigenspace $H_w^N = \text{span}\{|n\rangle, n \in \mathbf{Z}_M\}$, and its dual *angle operator* $\Theta |\vartheta_n\rangle = \vartheta_n |\vartheta_n\rangle$, with its eigenspace $H_w^{\Theta} = \text{span}\{|\vartheta_n\rangle, \vartheta_n \in \frac{1}{2\pi}\mathbf{Z}_M\}$, related by a finite Fourier transform with H_w^N . Two ARWs and its associated QRWs (modulo local unitary operators in coin spaces, as explained above), can be constructed: the *action* random walk on \mathbf{Z}_M , with CPTP map constructed with Kraus generators been the step operators in the action operator eigenstates i.e $S_{\pm} |n\rangle \equiv e^{\pm i\Theta} |n\rangle = h^{\pm 1} |n\rangle = |n \pm 1\rangle$, and the *angle* random walk on $\frac{1}{2\pi}\mathbf{Z}_M$, with CPTP map constructed with Kraus generator been the step operators in the momentum operator eigenstates i.e $S_{\pm} |\vartheta_n\rangle \equiv g^{\pm 1} |\vartheta_n\rangle = e^{\pm iN} |\vartheta_n\rangle = |\vartheta_{n\pm 1}\rangle$;

iv) the Coherent State QRW ¹⁵[**CS**-QRW]: with the *annihilation operator* $a |\alpha\rangle = \alpha |\alpha\rangle$, with its eigenspace $H_w^a = \text{span}\{|\alpha\rangle, \alpha \in \mathbf{C}; \frac{d^2\alpha}{2\pi}\}$. Two ARWs and its associated QRWs (modulo local unitary operators in coin spaces, as explained above), can be constructed: the *annihilation* random walk on \mathbf{C} , with CPTP map constructed with Kraus generators been the step operators in the annihilation operator eigenstates i.e $S_{\pm} |\alpha\rangle \equiv e^{J(\pm 1)} |\alpha\rangle = D_{\pm 1} |\alpha\rangle = |\alpha \pm 1\rangle$. The step operators here identified as special case of the canonical coherent state displacement operator $D_{\pm\beta} = e^{\pm\beta a^\dagger \mp \bar{\beta} a} = e^{J(\pm\beta)}$, have based the indicated step property on the following operator identity $D_\alpha D_\beta = e^{-i\alpha \times \beta} D_{\alpha+\beta}$, applied for the case of co-linear α, β vector on complex plane.

To give an explicit identification of the ARW based on the **hw** algebra constructed in ¹⁵, as a quantum random walk, and in particular as a **CS**-QRW, we make the following choices: the transition probabilities are

$p_+ = p, p_- = 1 - p$, the coin state is $\rho_c = |0\rangle\langle 0|$, the U operator is $U(p, 1 - p)$, and the step operators are CS displacement operators with steps $\pm\beta \in \mathbf{C}$. Then the total 1-step evolution operator in the coin+walker system is $V = \sum_{m=\pm} P_m U \otimes S_m = \begin{pmatrix} \sqrt{p}D_{+\beta} & \sqrt{1-p}D_{+\beta} \\ \sqrt{1-p}D_{-\beta} & -\sqrt{p}D_{-\beta} \end{pmatrix}$, and the reduced walker evolves in 1-step by the CPTP $\varepsilon_V(\varrho_w) = pD_{+\beta}\varrho_w D_{+\beta}^\dagger + (1-p)D_{-\beta}\varrho_w D_{-\beta}^\dagger = \text{Tr}_c(V\varrho_c \otimes \varrho_w V^\dagger)$.

For n steps the evolution of the walker has been chosen in ¹⁵ to be $\varepsilon_V^n(\varrho_w)$. It is important to notice that this is a choice based on the ARW construction methodology, and that our present treatment of the same walk as a QRW, sees the $\varepsilon_V^n(\varrho_w)$ type of evolution to result from a partial tracing of the coin system at every step. Our previous discussion of other types of tracing schemes motivates the study of **CS-QRWs** with delayed tracing, in order to investigate phenomena such as enhanced or anomalous diffusion in ARWs. This problem will be taken up elsewhere.

5. Discussion

We have outlined a mathematical framework where the conception of random walk and its associated statistical notions, and equations of motion, can both be studied in an algebraic and quantum mechanical manner. ARWs and QRWs appear to be two aspects of the same mathematical device, so their interconnection serves to conceptually clarify the common ground between them and to enrich the heuristics of formulating new problems and methodically searching for their solutions.

Quantum random walks are important both as quantum algorithms to experimentally be realized and as modules in a general quantum computing algorithm-devise that could outperform some classical rival. The connection ARW-QRW could serve to generalize, unify and compare such algorithms.

Also Quantum Information Processing concepts and tools, could be developed for ARW-QRWs. The step taken here is only a preliminary one towards developing such a theory.

Finally, ARW-QRWs come with lots of free choices for its constituting parameters. To mention only one expected application in the field of Open Quantum Systems, we should emphasize the importance of choosing the functional in e.g the **hw** ARW. Various choices of functionals in terms of types of coherent state vectors, combined together with various choices of ordering the operator basis in the enveloping algebra $\mathcal{U}(\mathbf{hw})$, i.e normal, antinormal, symmetric etc, could serve as a guiding rule for constructing

quantum master equations for open boson systems interacting with various types of quantum mechanical baths.

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